*Journal of Statistical Physics, Vol. 87, Nos. 3/4, 1997*

# Coulomb Interaction Symmetries and the Mayer Series in the Two-Dimensional Dipole Gas

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*Received July 17, 1996*

We study the Mayer series of the two-dimensional dipole gas in the high-temperature, low-density regime. Without performing any multiscale analysis, we obtain bounds showing that the Mayer coefficients are finite in the thermodynamic limit. These bounds are obtained by exploiting a particular partial symmetry of the interaction (which we name *θ-symmetry*), already used in some problems related to the two-dimensional Coulomb gas. By direct bounds on some Mayer graphs we also conjecture that any technique based uniquely on the θ-symmetry will not be sufficient to prove analyticity of the series.

**KEY WORDS:** Coulomb interaction symmetries; Mayer series; two-dimensional dipole gas.

## 1. INTRODUCTION

The dipole gas and other gases of particles interacting through Coulomb forces are statistical systems that have been studied for a long time. In particular, for the dipole gas, the lack of screening is well known[9] and the analyticity of the pressure in the high-temperature and low-activity region has been shown, in an indirect way, by mapping the system onto a field theory (sine-Gordon transformation) and using renormalization group methods (see ref. 7 for the $d \geqslant 3$ lattice model, and ref. 1 for the $d \geqslant 1$ continuum one). A direct proof of the analyticity of the pressure (e.g., for the two-dimensional case) based on bounds for the coefficients of the Mayer series is still an open problem. The close relationship between this

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0022-4715/97/0500-0877$12.50/0 

model and the Coulomb gas in the Kosterlitz–Thouless phase ($\beta > 8\pi$), allied to the nonexistence of any proof for the analyticity of the pressure in the Coulomb gas (indirect arguments are presented in refs. 4, 2, and 8), makes this problem even more interesting.

The Mayer series for the case of the Coulomb gas with $\beta > 8\pi$ was studied directly in ref. 5 and later in ref. 10. Both works use multiscale analysis and exploit one certain symmetry of the model in order to prove that the coefficients of the series are finite in the thermodynamic limit.

In the present paper, in an attempt to understand the role played by symmetries in systems with Coulomb interactions, we study the two-dimensional dipole gas. Exploiting the same symmetry (already used in refs. 5 and 10), we prove, *without recourse to multiscale analysis*, finiteness of the Mayer series coefficients in the thermodynamic limit. We argue, however, that a richer symmetry structure (i.e., the total symmetry of the dipole gas) may be necessary to prove directly the analyticity of the pressure.

We introduce now the model and its properties and make more precise some facts stated above. Throughout this article generic constants are denoted by $C$.

The dipole gas is a system of classical particles in a two-dimensional box $\Lambda$ carrying vectorial moments $q$ of unit length ($q \in \mathbf{R}^2$, $q^2 = 1$), interacting via the potential

$$V(\xi_1, \xi_2) = (q_1 \cdot \partial_{x_1})(q_2 \cdot \partial_{x_2})\, W(x_1 - x_2)$$
$$= \int \frac{dp}{(2\pi)^2} \frac{(q_1 \cdot p)(q_2 \cdot p)}{p^2}\, e^{-p^2} e^{ip \cdot (x_1 - x_2)} \tag{1.1}$$

where $\xi = (x, q)$, $x \in \Lambda \subset \mathbf{R}^2$ is the particle position, $q \in S^1$ is the dipole moment, and $W$ is the Coulomb potential

$$W(x_1 - x_2) = \int \frac{dp}{(2\pi)^2} [e^{ip \cdot (x_1 - x_2)} - 1] \frac{e^{-p^2}}{p^2}$$
$$= \frac{1}{4\pi} \int_1^{\infty} \frac{d\alpha}{\alpha} [e^{-(x_1 - x_2)^2/4\alpha} - 1] \tag{1.2}$$

We use in (1.2) the same UV cutoff as in ref. 5. We may write (1.1) as

$$V(\xi_1, \xi_2) = V(x_1 - x_2, \theta_1, \theta_2)$$
$$= \frac{1}{8\pi} [\cos(\theta_1 - \theta_2)\, e^{-(x_1 - x_2)^2/4}$$
$$+ \cos(\theta_1 + \theta_2 - 2\phi_{12})\, f(x_1 - x_2)] \tag{1.3}$$

where $\theta_i$ is the angle between $q_i$ and some fixed axis, $\phi_{12}$ is the angle between $x_1 - x_2$ and the same axis, and

$$f(x_1 - x_2) = e^{-(x_1 - x_2)^2/4} - \frac{1 - e^{-(x_1 - x_2)^2/4}}{(x_1 - x_2)^2/4} \tag{1.4}$$

Note that $V(\xi_1, \xi_2) \sim O(1/|x_1 - x_2|^2)$ as $|x_1 - x_2| \to \infty$, and also that $V(\xi, \xi) = 1/(8\pi)$.

There are several symmetries in $V_{ij} \equiv V(\xi_i, \xi_j)$. For example, $V(x_i - x_j, \theta_i + \pi, \theta_j) = -V(x_i - x_j, \theta_i, \theta_j)$, which we name the *θ-symmetry*, is the one used later to prove finiteness of the Mayer series coefficients. It implies that the interaction averaged over dipole orientations vanishes. Another symmetry (say, *φ-symmetry*) involving $\phi_{ij}$, the angular variable of $x_i - x_j$, makes finite the integral of the potential over a sphere: $\int dx_j\, V_{ij} < \infty$; the potential, however, is not absolutely integrable (i.e., it is not $L_1$); more comments are given in Section 3. Note that the second term in (1.3), responsible for the slow decay of $V$ (i.e., the delicate term), depends on $\cos(\theta_i + \theta_j - 2\phi_{ij})$ with a rich symmetry (involving the $\theta$ and $\phi$ symmetries).

Note also that $V$ is a stable potential, which may be verified by using (1.1) and checking that $\sum_{i=1}^N \sum_{j=1}^N V_{ij} \geqslant 0$, $\forall N$, $\xi_1, \ldots, \xi_N$.

The partition function for the dipole gas in the grand canonical ensemble is

$$Z_\Lambda(\beta, \lambda) = \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} \int d\xi_1 \cdots \int d\xi_N \exp\left(-\beta \sum_{i<j} V_{ij}\right) \tag{1.5}$$

where $\int d\xi_i = \int_0^{2\pi} d\theta_i \int_\Lambda dx_i$, $\lambda$ is the activity, and $\beta$ is the inverse temperature. The pressure is the thermodynamic limit $|\Lambda| \to \infty$ of

$$\beta p_\Lambda(\lambda, \beta) = \frac{1}{|\Lambda|} \log Z_\Lambda(\beta, \lambda) \tag{1.6}$$

The Mayer series for the pressure is a formal expansion in powers of $\lambda$ obtained by writing $\exp(-\beta V_{ij}) = [\exp(-\beta V_{ij}) - 1] + 1$, then expanding $\prod_{i<j} \exp(-\beta V_{ij})$ as a sum of terms labeled by Mayer graphs, and taking the logarithm (which eliminates the unconnected graphs).[6] We have

$$\beta p_\Lambda(\lambda, \beta) = \sum_{N=1}^{\infty} C_{N,\Lambda}(\beta)\, \lambda^N \tag{1.7}$$

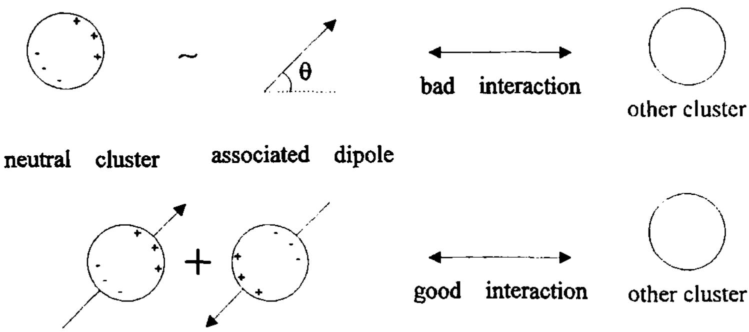


Fig. 1.

with

$$C_{N,\Lambda}(\beta)=\frac{1}{N!\,|\Lambda|}\int d\xi_1\cdots d\xi_N \sum_{G\in\mathscr{G}_N}\prod_{(i,j)\in G}[e^{-\beta V_{ij}}-1] \tag{1.8}$$

where $\mathscr{G}_N$ is the set of all connected Mayer graphs between $N$ points.

A direct proof for the analyticity of the pressure as a function of $\lambda$ in the thermodynamic limit means finding a bound $|C_{N,\Lambda}(\beta)|<[C(\beta)]^N$ uniform in $\Lambda$. As $V_{ij}$ is not $L_1$, standard cluster expansion arguments fail[6] and even the proof of finiteness of the coefficients become nontrivial.

Finiteness of the Mayer series coefficients in the two-dimensional Coulomb gas above $8\pi$ has been directly proved in refs. 5 and 10, as mentioned. A multiscale analysis is performed there, describing the gas as a system of clusters of hierarchically organized charges, and a symmetry property is used to solve problems in the thermodynamic limit arising in the analysis of the interaction between neutral clusters of charges (dipoles, basically). Precisely, in ref. 5 a necessary gain in the decay of the interaction between neutral clusters is obtained by summing two neutral clusters with the charge distribution reversed (i.e., flipping dipoles); see Fig. 1.

In ref. 10 the same gain is obtained by integrating over $\theta$ (i.e., rotating dipoles; see Fig. 2.

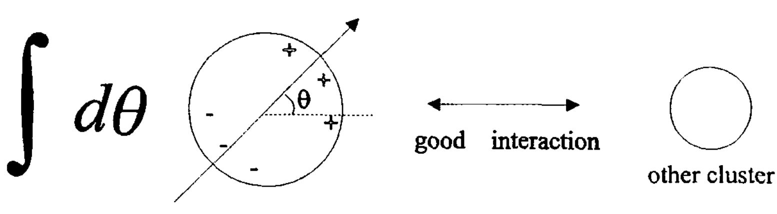


Fig. 2.

Notice that both procedures essentially involve the use of what we called the $\theta$-symmetry of the dipole interaction.

In the next section, we use the $\theta$-symmetry to prove that the coefficients of the Mayer series for the pressure of the bidimensional dipole gas are finite in the thermodynamic limit. Section 3 is devoted to arguing that the full symmetry of the dipole gas may be necessary in order to prove analyticity of the pressure. In Section 4 we make final comments.

## 2. FINITENESS OF THE MAYER SERIES COEFFICIENTS

As mentioned before, since the dipole potential (1.1) is not $L_1$, it is nontrivial to show that the $N$th-order Mayer coefficient for the pressure, given by (1.8), is finite in the thermodynamic limit. Recall that it is expressed as a sum of terms labeled by connected Mayer graphs on $N$ points, namely

$$C_{N,\Lambda}(\beta)=\frac{1}{N!}\sum_{G\in\mathscr{G}_N} C^G_{N,\Lambda}(\beta) \tag{2.1}$$

where

$$C^G_{N,\Lambda}(\beta)=\frac{1}{|\Lambda|}\int_\Lambda dx_1\cdots\int_\Lambda dx_N\int_0^{2\pi} d\theta_1\cdots\int_0^{2\pi} d\theta_N \prod_{(i,j)\in G}[e^{-\beta\sigma_i\sigma_j V_{ij}}-1] \tag{2.2}$$

Therefore, we show now, by using only the $\theta$-symmetry of the model, that we may obtain bounds for the thermodynamic limit $|\Lambda|\to\infty$ of $C^G_{N,\Lambda}(\beta)$.

Our results are stated as follows.

**Proposition 1.** Let $V_{ij}=V(x_i-x_j,\theta_i,\theta_j)$ be a two-dimensional potential, with $x_i-x_j\in\mathbf{R}^2$, $\theta_i,\theta_j\in[0,2\pi]$, and such that:

1. $|V(x_i-x_j,\theta_i,\theta_j)|\leqslant C/[1+(x_i-x_j)^2]$
2. $V(x_i-x_j,\theta_i+\pi,\theta_j)=V(x_i-x_j,\theta_i,\theta_j+\pi)=-V(x_i-x_j,\theta_i,\theta_j)$

Then, for all $G\in\mathscr{G}_N$, $C^G_{N,\Lambda}(\beta)$ as in (2.2) admits a bound uniform in $\Lambda$.

To establish the proposition, besides an adequate use of the $\theta$-symmetry (property 2 above) we need the following lemma (whose proof is obtained following the demonstration of Lemma 2, next section).

**Lemma 1.** Let $\Sigma$ be the ball of radius $R$ centered in the origin (then $|\Sigma| = \pi R^2$). Let $V$ satisfy part 1 of the proposition above. Then, $\forall x, y \in \mathbf{R}^2$,

$$\lim_{R \to \infty} \left\{ \int_\Sigma dx_1 \cdots \int_\Sigma dx_N \, |V(x - x_1)| \right.$$

$$\left. \times |V(x_1 - x_2)| \cdots |V(x_{N-1} - x_N)| \cdot |V(x_N - y)| \right\} < C_N < \infty$$

i.e.,

$$\underbrace{|V| * \cdots * |V|}_{N+1} (x - y)$$

[the $(N+1)$-fold convolution] is finite in the thermodynamic limit.

Note that the lemma alone takes care of all Mayer graphs which cannot be separated into two subgraphs by cutting a single line (this kind of graph is termed *one-particle irreducible*[6]). In fact, recall that each line $(i, j)$ in the graph represents a function $v_{ij} = \exp(-\beta V_{ij}) - 1 \approx \beta V_{ij}$ satisfying the hypothesis of the proposition. Thus we may apply repetitively the lemma in order to bound these graphs. For example, the first graph in Fig. 3 is bounded by $C\,|v| * |v| * |v|\,(0)$, whereas the second is bounded by noting that $x_7$ may be integrated out, $\int dx_7 \, |v(x_1 - x_7)\, v(x_7 - x_4)| = |v| * |v|\,(x_1 - x_4) < C$, and that the loop 1-2-3-4-5-6-1 is bounded as in graph 1.

Clearly there are graphs which are not controlled by Lemma 1; e.g., see Fig. 4. These graphs are precisely the ones which separate into two disconnected components by cutting just one line, i.e., the one-particle *reducible* ones. The special lines which disconnect the graph when cut are called *tree lines*. In order to control these graphs we make explicit the $\theta$-symmetry by introducing new variables (the charges) $\sigma_i = \pm 1$ and noting


Fig. 3.

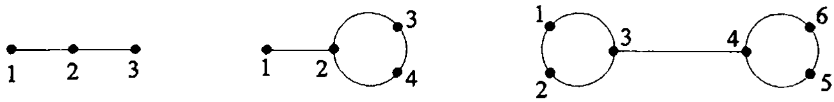


Fig. 4.

that for any function $F$ of the potential $V$ satisfying condition 2 of the proposition we may write

$$\int_0^{2\pi}\int_0^{2\pi} F[V(x_i - x_j, \theta_i, \theta_j)]\, d\theta_i\, d\theta_j$$

$$= \sum_{\sigma_i, \sigma_j = \pm 1} \int_0^{\pi}\int_0^{\pi} F[\sigma_i \sigma_j V(x_i - x_j, \theta_i, \theta_j)]\, d\theta_i\, d\theta_j$$

Hence, for any Mayer graph $G$, (2.2) becomes

$$C^G_{N,\Lambda}(\beta) = \frac{1}{|\Lambda|} \sum_{\sigma_1, \ldots, \sigma_N = \pm 1} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \int_0^{\pi} d\theta_1 \cdots \int_0^{\pi} d\theta_N \times \prod_{(i,j) \in G} [e^{-\beta\sigma_i\sigma_j V_{ij}} - 1] \tag{2.3}$$

We proceed by writing

$$\exp(-\beta\sigma_i\sigma_j V_{ij}) - 1 = -\sigma_i\sigma_j \sinh(\beta V_{ij}) + \cosh(\beta V_{ij}) - 1$$

graphically described as the split in Fig. 5. The name "soft" comes from the fact that $\sinh(\beta V_{ij})$ is of order $\beta V_{ij}$ ($\beta$ and $V_{ij}$ small), hence its volume integration is delicate, and "hard" because $\cosh(\beta V_{ij}) - 1$ is of order $(\beta V_{ij})^2$, an $L_1$ function.

With the split above, the graph $G$ [and so $C^G_{N,\Lambda}(\beta)$ in (2.2)] becomes the sum of $2^{|G|}$ terms ($|G|$ is the number of links in $G$), each of these terms being represented by a graph $G'$ with the same topological structure as $G$ [i.e., $G'$ has a line between a pair $(i, j)$ whenever $G$ has this line], but now

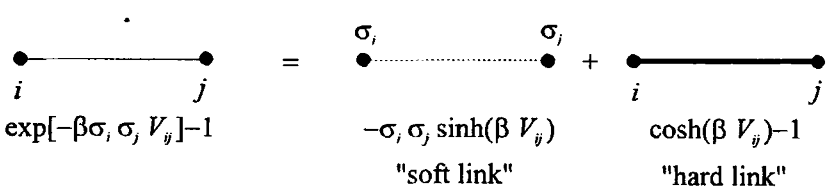


Fig. 5.

the links of $G'$ are either soft or hard as in Fig. 5. The key point is that these new graphs $G'$ have associated charges: each soft link reaching the vertex $i$ carries a charge $\sigma_i$. When we sum over the charges, it follows that all graphs $G'$ with an odd number of soft links reaching any vertex $i$ vanish [since $\sum_{\sigma_i=\pm 1} (\sigma_i)^k = 0$ if $k$ is odd]. It is not difficult to see that only those $G'$ survive in which the soft lines appear in one-particle irreducible subgraphs. In other words, any tree line in $G'$ must be hard.

For example, among the graphs of Fig. 4 the only surviving $G'$ graphs are those shown in Fig. 6.

By using Lemma 1, we may integrate safely soft-line subgraphs, the remaining hard links being easily taken into account by the fact that they represent an integrable factor. It follows that any $G'$ is uniformly bounded in $\Lambda$, and hence $G$ is uniformly bounded in $\Lambda$, proving the proposition.

**Remark.** Although the splitting of Fig. 5 leads to a proliferation of terms, i.e., a single graph $G$ proliferates into $2^{|G|}$ new terms $G'$, it can be shown that, by using the $\theta$-symmetry, it is possible to rewrite $C^G_{N,\Lambda}$ of (2.2) as one single term

$$C^G_{N,\Lambda}(\beta) = \frac{1}{|\Lambda|} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{(i,j)\in G} B_{ij} \qquad (2.4)$$

where

$$B_{ij} = \begin{cases} \cosh(\beta V_{ij}) - 1 & \text{if } (i,j) \text{ is a tree line} \\ e^{-\beta V_{ij}} - 1 & \text{otherwise} \end{cases}$$

Graphically, this means that in each graph $G$ we can replace $e^{-\beta V_{ij}} - 1$ by $\cosh(\beta V_{ij}) - 1$ whenever $(i, j)$ is a tree-line without affecting the value $C^G_{N,\Lambda}$; e.g., see Fig. 7. Actually this symmetry seems to have an even greater

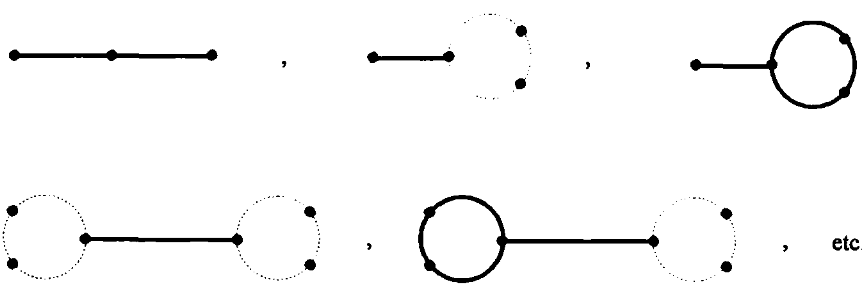

Fig. 6.

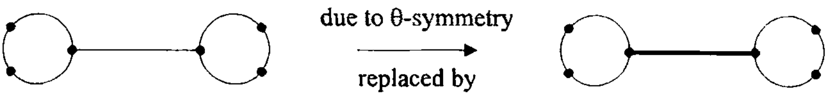


Fig. 7.

range of applicability. For example, the second graph 2 in Fig. 3, already controlled in the thermodynamic limit by Lemma 1 alone, has its bound improved by noting that vertices 1 and 4 cannot have three soft links (at least one must be hard).

## 3. THE SYMMETRIES AND THE ANALYTICITY PROBLEM

As we have seen in the previous section, by just considering the $\theta$-symmetry present in the dipole gas potential, we were able to show that the series coefficients remain finite in the thermodynamic limit. We point out again that this symmetry (together with a multiscale formalism) has been the mechanism used to prove also the existence of the formal Mayer series of the Coulomb gas above $8\pi$.[5,10] We believe, however, that using only the "$\theta$-symmetry" we will not be able to show the analyticity of the dipole gas pressure: in this section we present some arguments which lead us to this conjecture.

To make explicit the extent of each symmetry, we compare Mayer graphs of two different potentials: the dipole gas potential (1.3) and another one with a similar decay ($1/x^2$ as $x \to \infty$), but containing only the "$\theta$-symmetry." We take $W_{ij}^{(\alpha)} = \sigma_i \sigma_j V_{ij}^{(\alpha)}$, $\alpha = 1, 2$, with

$$V_{ij}^{(1)} = V(x_i - x_j, \theta_i, \theta_j) \qquad \text{as in (1.3)} \tag{3.1}$$

and

$$V_{ij}^{(2)} = \frac{1}{1 + |x_i - x_j|^2} \tag{3.2}$$

where $\sigma_i, \sigma_j = \pm 1$ as already defined; $x_i, x_j \in \Lambda$; $\theta_i, \theta_j \in [0, \pi]$. Remark that any technique using only the $\theta$-symmetry means taking into account $|V^{(1)}|$ (instead of $V^{(1)}$) when bounding the integrals related to the Mayer graphs. The effect of the modulus is to turn off the angular symmetries of the dipole potential (apart from the $\theta$-symmetry). Thus, actually, we should compare $V^{(1)}$ with $|V^{(1)}|$. However, we take $V^{(2)}$ above for the sake of simplicity: it leads to simpler calculations (details below) and essentially behaves as $|V^{(1)}|$. More precisely, Lemma 3 below clearly remains true for $|V^{(1)}|$ replacing $V^{(2)}$.

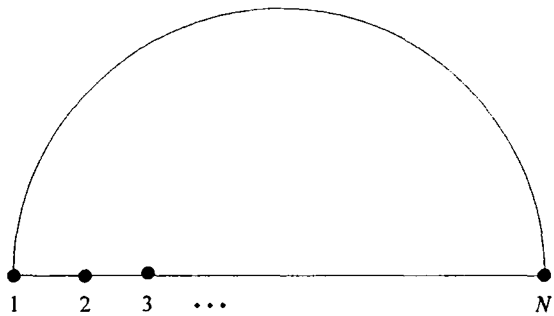

Fig. 8.

As described in the previous section, due to the sum over charges $\sigma_i$ (i.e., the "$\theta$-symmetry"), we expect Mayer series with finite coefficients for both potentials. In order to show analyticity of the pressure we must bound by $C^N$ the Mayer coefficients, which are given by a sum of several positive and negative graphs. The control of these signs, related to the potential stability, will not be carried out here. We will only examine the behavior of certain graphs with $N$ to show that for the dipole potential $W_{ij}^{(1)}$ (with the full symmetry) they are bounded above by $C^N$, whereas for the simplified potential $W_{ij}^{(2)}$ these graphs have $N!\ C^N$ as a lower bound.

Our claim is stated in the following lemma (whose proof essentially works also for Lemma 1; comments below).

**Lemma 2.** Let $G$ be the loop graph of Fig. 8 with $N$ lines and $N$ vertices, and let $\Lambda$ be the ball of radius $R$ centered at the origin (then $|\Sigma| = \pi R^2$). Then we have for

$$\mathscr{G}_N^{(\alpha)} \equiv \lim_{R \to \infty} \frac{2^N}{|\Sigma|} \int d\xi_1 \cdots \int d\xi_N \prod_{(i,j) \in G} V_{ij}^{(\alpha)}, \qquad \alpha = 1, 2$$

with $V^{(1)}$, $V^{(2)}$ as defined above, that:

1. $|\mathscr{G}_N^{(1)}| \leqslant C^N$.
2. $|\mathscr{G}_N^{(2)}| \geqslant N!\ C^N$.

*Proof.* Part 2: We note that $V^{(2)}(x) = 1/(1+x^2)$ is in $L^2(\mathbf{R}^2)$, and its Fourier transform $\tilde{V}^{(2)}(p)$ is asymptotically given by[6]

$$\begin{aligned} \tilde{V}^{(2)}(p) &\approx -\frac{1}{2\pi} \ln |p|, && \text{small } |p| \\ \tilde{V}^{(2)}(p) &\approx \left(\frac{1}{8\pi\,|p|}\right)^{1/2} \exp(-|p|), && \text{large } |p| \end{aligned} \tag{3.3}$$

It is easy to check that $[\tilde{V}^{(2)}(p)]^N$ is also in $L^2(\mathbf{R}^2)$. From the Parseval theorem for the Fourier transform we have

$$\int [\tilde{V}^{(2)}(p)]^2 e^{ip\cdot x}\,dp = \int V^{(2)}(y)\,V^{(2)}(x-y)\,dy$$

and iterating it, we have

$$\underbrace{V^{(2)} * V^{(2)} * \cdots * V^{(2)}}_{N \text{ times}}(x) = \int [\tilde{V}^{(2)}(p)]^N e^{ip\cdot x}\,dp \qquad (3.4)$$

from which it follows that the convolution $V^{(2)} * V^{(2)} * \cdots * V^{(2)}$ is in $L^2(\mathbf{R}^2)$. Using the dominated convergence theorem, $\mathscr{G}_N^{(2)}$ is given explicitly by

$$\begin{aligned}\mathscr{G}_N^{(2)} &= \lim_{R\to\infty} \frac{2\pi^N}{|A|} \int dx_1 \cdots dx_N \frac{1}{1+(x_1-x_2)^2}\frac{1}{1+(x_2-x_3)^2}\cdots \\ &\quad\times \frac{1}{1+(x_{N-1}-x_N)^2}\frac{1}{(1+(x_N-x_1)^2} \\ &= \underbrace{V^{(2)} * V^{(2)} * \cdots * V^{(2)}}_{N \text{ times}}(0) \qquad (3.5)\end{aligned}$$

and thus we have, for $x=0$, $\mathscr{G}_N^{(2)} < \infty$. Note that with (3.4) we also prove Lemma 1. And from (3.3) above

$$\begin{aligned}\mathscr{G}_N^{(2)} &= \int_0^\infty \rho\,d\rho \int_0^{2\pi} d\varphi\,[V^{(2)}(\rho\cos\varphi, \rho\sin\varphi)]^N \\ &\geqslant C^N \int_0^1 d\rho\,\rho\,|\ln \rho|^N \geqslant C^N N! \qquad (3.6)\end{aligned}$$

proving part 2.

Part 1 follows in a similar manner using

$$V^{(1)}(\theta_i, \theta_j, x_i - x_j) = \int dp \frac{(d_i\cdot p)(d_j\cdot p)}{p^2} e^{-p^2} e^{ip\cdot(x_i-x_j)} \qquad (3.7)$$

where $d_i, d_j$ are unit vectors describing the dipoles at $x_i$ and $x_j$ $(d_i\cdot e_1 = \cos\theta_i)$ and noting that $\tilde{V}^{(1)} = [(d_i\cdot p)(d_j\cdot p)/p^2]\exp(-p^2)$ is bounded in every neighborhood of $p=0$. Remark that this bound is an effect of the $\phi$-symmetry.

We stress once more that $W^{(2)}$ is "morally" $W^{(1)}$ with the $\theta$-symmetry only, i.e., $W^{(1)}$ with the other angular symmetries turned off; moreover, the comparison between the dipole potential $W^{(1)}$ and the potential $W^{(2)}$ makes sense only in two dimensions. The expression (3.2) describes a potential with the $\theta$-symmetry and with the *same* decay of the dipole gas only for $d=2$. In one dimension, for example, the analogous dipole gas (3) would be a Gaussian (i.e., a smooth function exponentially decaying at large distances) which is the smooth UV cutoff version of the delta function, the one-dimensional dipole potential. This potential has a poorer symmetry (compared with the two-dimensional case), but it is obviously absolutely integrable and so the Mayer series trivially converges.

## 4. FINAL COMMENTS

Another argument for the necessity of using more than the $\theta$-symmetry to prove the analyticity of the pressure appears when we turn to the well-known proof based on renormalization group techniques.

With the sine-Gordon transformation mapping the statistical system on a field theory problem, the dipole partition function becomes

$$\mathscr{Z}_\Lambda = \int P(d\phi) \exp\left\{\lambda \int_\Lambda dx \int_0^{2\pi} d\theta \exp[i\beta^{1/2}(q\cdot\partial_x)\,\phi(x)]\right\}$$

where, roughly, $P(d\phi)$ is a measure on $\phi$ with inverse Laplacian as covariance (see ref. 1 or ref. 5 for details). Due to the $\theta$-symmetry we have $q(\theta+\pi) = -q(\theta)$, and so

$$\mathscr{Z}_\Lambda = \int P(d\phi) \exp\left\{2\lambda \int_\Lambda dx \int_0^{\pi} d\theta \cos[\beta^{1/2}(q\cdot\partial_x)\,\phi(x)]\right\}$$

That is, the $\theta$-symmetry is responsible for the even-function cosine in the field potential. It is important, but not the main point: in ref. 7, for example, $\cos(\partial\phi)$ and $(\partial\phi)^4$ are treated at the same time as a class of functions leading to the same infrared behavior. The crucial fact in the field theory problem is that the action is given by a Laplacian plus a perturbation depending only on $\partial\phi$ [which is related to an interaction such $\partial_i\partial_j/\partial^2$ as in (1.1)]. Dependence only on $\partial\phi$ is due to the full symmetry and leads to a renormalization group with only marginal and irrelevant terms.

## ACKNOWLEDGMENTS

This work was partially supported by CNPq and FAPEMIG (Brazil).

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